been applied to many hydrates (El Saffar, 1966, 1968). Some of these hydrates have since been studied by neutron diffraction. Detailed comparison shows that the experimental vectors obtained by n.m.r. are in rather good agreement with the corresponding neutron-diffraction results. The consistency of the p–p vectors given in Table 1 with the H-bonding scheme suggested by the X-ray authors, verifies the correctness of the H-bonding scheme and testifies to the validity of the n.m.r. method.

The conflict between the n.m.r. results reported here and those of Visweswaramurthy (1963) may be resolved by modifying Visweswaramurthy's Fig. 2 from which he obtained all his results. As the Figure stands, the Pake curves shown (c axis rotation) do not satisfy the point-group symmetry since those curves are not symmetrical about the a axis. The point-group requirement is satisfied provided the Pake curve labelled 2 is discarded and the Φ scale is shifted so that a is at $\Phi = 20$ or $-30°$. If the shift $-30°$ is accepted, the Pake curves labelled 1 and 3 in Visweswaramurthy's Fig. 2 conform with the results reported here.

Distances and angles relating to the hydrogen bonds are given in Table 3. It is interesting to note a correspondence between the OH---X distances and the related $\angle$O–H---X for $MgS_2O_3.6H_2O$. The shortest OH---S bond (3·195 Å) and the shortest OH---O bond (2·658 Å) belong to H_2O_{II}. Equal 'bending' of these bonds ($\angle$O–H---$X = 8°$) may be due to the fact that the OH---S and OH---O interactions in this case are of equal strength. The same may be said about the interactions relating to H_2O_I. The angles and distances relating to H_2O_{III} given in Table 3 agree with the general observation that shorter hydrogen bonds tend to deviate less from linearity than longer ones (Hamilton, 1962). This tendency has been recently explained (Chidambaram & Sikka, 1968) on the basis of a modified Lippincott–Schroeder potential function for the hydrogen bond.

Hydrogen bonds of the type OH---S, have not been studied, to the author's knowledge, by neutron diffraction. It is, therefore, considered useful to give the angles and distances relating to OH---S bonds found in $Na_2S_2O_3.5H_2O$. This compound was investigated by Taylor & Beevers (1952) with the use of X-rays, and by Murty & El Saffar (1962) by the use of n.m.r.

References

CHIDAMBARAM, R. & SIKKA, S. K. (1968). *Chem. Phys. Lett.*
EL SAFFAR, Z. M. (1966). *J. Chem. Phys.* **45**, 4643.
EL SAFFAR, Z. M. (1968). *Acta Cryst.* B**24**, 1131.
HAMILTON, W. C. (1962). *Ann. Rev. Phys. Chem.* **13**, 19.
HOLCOMB, D. F. & PEDERSEN, B. (1963). *J. Chem. Phys.* **38**, 51.
MURTY, C. R. K. & EL SAFFAR, Z. M. (1962). *Acta Cryst.* **15**, 536.
NARDELLI, M., FAVA, G. & GIRALDI, G. (1962). *Acta Cryst.* **15**, 227.
PAKE, G. E. (1948). *J. Chem. Phys.* **16**, 327.
TAYLOR, P. G. & BEEVERS, C. A. (1952). *Acta Cryst.* **5**, 341.
VISWESWARAMURTHY, S. (1963). *Acta Cryst.* **16**, 933.

Acta Cryst. (1969). B**25**, 1711

The Geometrical Basis of Crystal Chemistry. X. Further Study of Three-Dimensional Polyhedra.

BY A. F. WELLS

Chemistry Department and Institute of Materials Science, University of Connecticut, Storrs, Connecticut 06268, *U.S.A.*

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A further study is made of the three-dimensional polyhedra of part VII which conform to the equation

$$\Sigma f_n[4-(2-n)(2-p)] = 2p(2-t)$$

for tessellations (n,p), where t is the number of tunnels connecting each repeat unit to its neighbours. An investigation is made of the relation between a 3-D polyhedron and its complement, (*i.e.* the space not occupied by the polyhedron) and of those polyhedra which can be constructed with plane equilateral triangular faces, the 3-D homologues of the triangulated Platonic solids.

In part VII (Wells & Sharpe, 1963) a study was made of tessellations (n,p) for which p exceeds the highest value attainable in a plane net. For example, in the series $(3,p)$ the cases $p = 3, 4$, and 5 correspond to convex polyhedra, and $p = 6$ to the plane triangulated net. It was shown that certain tessellations $(3,p)$ with $p \geq 7$ may be inscribed on infinite surfaces which arise by uniformly inflating the links of 3-dimensional nets to form systems of tunnels which meet at the points of the original net. The links of the tessellations are geodesics on surfaces of continuously varying curvature. The models described in part VII were made with (curved) links of equal length. These 3-D polyhedra may alternatively be described as built of polyhedral units containing Z points placed at the points of a 3-D net and joined through a number (t) of tunnels to identical units situated at adjacent points of the net.

The families $(3,p)$ to $(6,p)$ inclusive were considered in turn for $t=3,4,5,6,8$, and 12, but for $n\geq 7$, only the systems $(n,3)$ (3-connected nets) were listed (Table 3). (For the sake of consistency the systems (5,6) for $t=10+4m$ and (7,4) for $t=8+3m$ should be included as having Z integral and $Z\geq t$, though it seems extremely unlikely that any of them could be realized. The system (10,3) with $t=5$ should be removed from this Table because its reciprocal has Z non-integral.) Examples were not given of all polyhedra, and in particular no detailed study was made of 12-tunnel systems. Because a number of new polyhedra have since been made and also because it is convenient to see at a glance which tessellations can be inscribed on a surface derived from a particular net, the information now available for triangulated polyhedra is summarized in Table 1. Repeat units of four new polyhedra are illustrated (as stereo pairs) in Figs. 1–3.

We now consider two aspects of these 3-D polyhedra. The first is the relation between the polyhedron and the space not occupied by the polyhedron, which may be described as the *complementary polyhedron*. The second concerns the geometrical shape of the polyhedra. They were illustrated in part VII as tessellations on curved surfaces, the polygons being equilateral but not equi-angular, since their edges are curved lines on surfaces of varying curvature. Certain of the polyhedra of Table 1 can be realized with plane equilateral (and hence equiangular) faces, p n-gons meeting at each point (vertex). Such bodies may be described as regular 3-D polyhedra, for they are the homologues of the five regular finite solids. Only the triangulated bodies will be considered in detail in this paper.

Complementary 3-D polyhedra

If a polygon is defined as a system of connected line segments, six types of regular polygon are recognized: digon, plane n-gon, finite skew polygon (lateral edges of antiprism), apeirogon (infinite straight line marked off in equal segments), plane zigzag, and helical polygon. Two of these define a plane and divide it into two parts, in one case equal (plane zigzag) and in the other unequal (plane n-gon). These polygons are 2-connected systems, finite or infinite. The p-connected tessellations ($p>2$) are inscribable on surfaces and may be described as polyhedra, of which the three simplest types are (a) finite (convex) polyhedra (b) infinite plane nets, and (c) infinite three-dimensional surfaces. These surfaces divide the whole of space into two parts. In (a) the two parts are unequal, in (b) they are equal, and we shall see that in (c) there may be equal or unequal division of space.

We now consider the relation between the two parts into which space is divided by the 3-D surfaces of part VII, the polyhedron and its complement. In all cases we shall suppose that the space not occupied by a 3-D polyhedron is also a 3-D polyhedron, *i.e.* any part of the polyhedron is accessible from any other (without passing through the surface), and that the same is true of the complementary polyhedron. This is not necessarily so, for although the 3-D polyhedron has this property the complementary space might consist of isolated volumes, finite, or infinite one- or two-dimensional. This is easily appreciated by visualizing the types of hole that may be made in a (supposedly infinite) solid block. The holes might be discrete finite holes, infinite one-dimensional (sets of unconnected tunnels), or infinite two-dimensional systems, equivalent to inflated 2-D nets arranged in parallel planes without connections between one net and another. Models of these three types are readily constructed from polyhedra such as hexagonal prisms or cubes. Since each 3-D polyhedron is based on a 3-D net (see Table 1) this is also true of complementary polyhedra satisfying the above criterion. The relation between a 3-D polyhedron and its complement is therefore the relation between certain pairs of interpenetrating 3-D nets.

It seems reasonable to distinguish degrees of interpenetration of 3D nets. If two cubic (10,3) nets, one left- and the other right-handed, interpenetrate [Fig. 4(a)], the connectedness of each point remains the same (3). If, however, two nets of the same hand interpenetrate, each point is equidistant from 6 others, though the helices of the two nets are still distinct [Fig. 4(b)]. Contrast the interpenetration of two diamond nets, when the helices intertwine around the same axes

Table 1. *Triangulated* 3-D *polyhedra*

[ ] indicates complementary polyhedron.

Net	t	(3,7)	(3,8)	(3,9)	(3,10)	(3,11)	(3,12)
Cubic (10,3)	3	VII.6(b)	VII.11	—	†	†	–
Diamond	4	X.17 (I_{4t})	VII.13 VII.12=X.14 (O_{4t})	X.1=X.13 (T_{4t})	–	†	–
NbO	4	—	VII.15	[I_{8t}]	–	†	[O_{8t}]
P	6	VII.6(c)	X.9(b) X.18 (I_{6t}) X.9(c)	VII.20(a)	VII.20(b) X.15 (O_{6t})	†	–
I	8	X.3	[VII.15] VII.18	X.2(a)=X.19 (I_{8t})	–	†	X.2(b)=X.16 (O_{8t})

† Z non-integral.

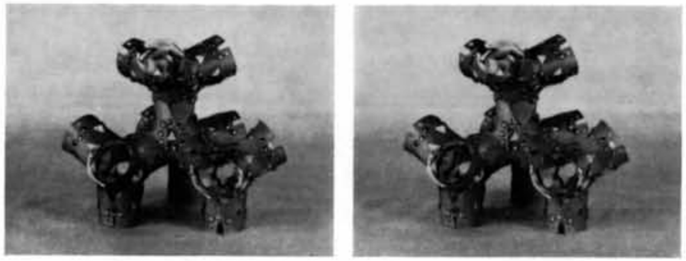
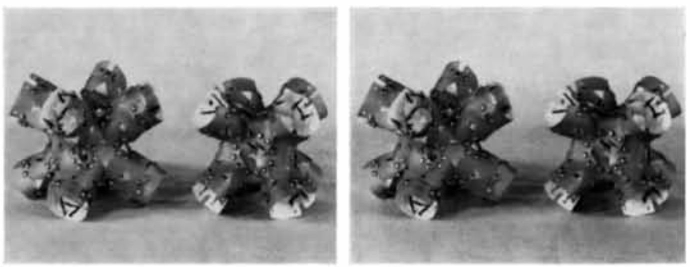


Fig. 1. Repeat unit of 4*t* (3,9). The same polyhedron built with plane equilateral faces is T_{4t} (Fig. 13).



(*a*) (*b*)

Fig. 2. Repeat units of (*a*) 8*t* (3,9) and (*b*) 8*t* (3,12). The same polyhedra built with plane equilateral faces are I_{8t} (Fig. 19) and O_{8t} (Fig. 16).

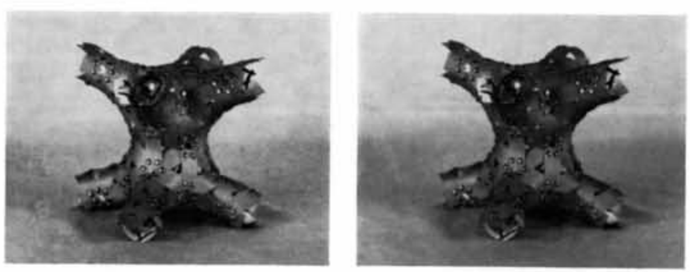


Fig. 3. Repeat unit of 8*t* (3,7).

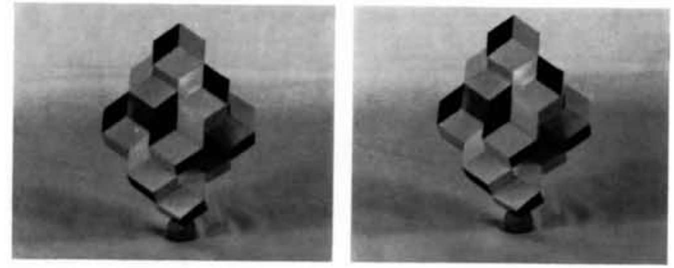


Fig. 8. Half-space-filling by rhombic dodecahedra.

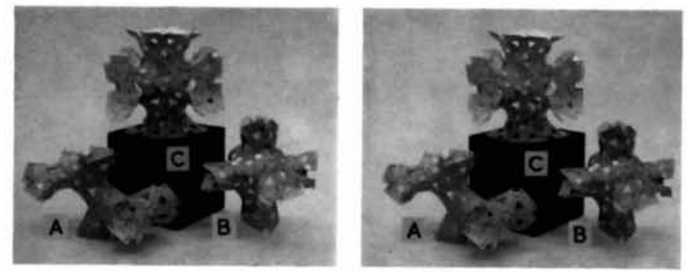


Fig. 9. Repeat units of polyhedra (3,8).

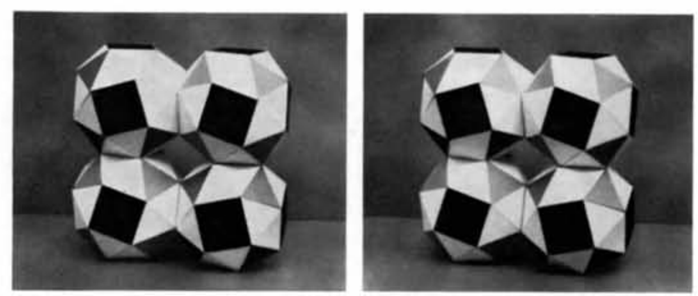


Fig. 10. Polyhedron (3,8) built from snub-cubes; the complementary polyhedron is I_{6t} (Fig. 18).

[Fig. 4(*c*)]. If two nets, *A* and *B*, interpenetrate to the maximum extent, so that each point of one net is surrounded symmetrically in three dimensions by points of the other net, we might expect each *A* (*B*) point in the composite system to be surrounded by (*a*) a tetrahedral group of 4*B* (*A*), (*b*) an octahedral group of 6*B* (*A*), or (*c*) a cubic group of 8*B* (*A*). Case (*a*) corresponds to the interpenetration of two diamond nets to form the *I* lattice (net), and (*c*) to the formation of the same net from two interpenetrating *P* lattices (nets). The second possibility, (*b*), where each *A* point is surrounded octahedrally by 6*B* and each *B* point octahedrally by 6*A*, corresponds to the formation of the NaCl structure from two interpenetrating *F* lattices. However, this pair of interpenetrating nets cannot represent a twelve-tunnel polyhedron and its complement for the following reason. Each *A* (*B*) point is connected to a number *p*, of *A* (*B*) points in its own net, by *A–A* (*B–B*) links. If *A–A* and *B–B* links intersect when the *A* and *B* nets interpenetrate, such nets cannot represent the relation between a 3-D polyhedron and its complement, for the points of intersection would lie within both the polyhedron and its complement. In the NaCl structure the lines representing the shortest Na–Na contacts intersect those representing the corresponding Cl–Cl contacts. There is, however, a pair of nets which can interpenetrate to give octahedral coordination of both *A* and *B*

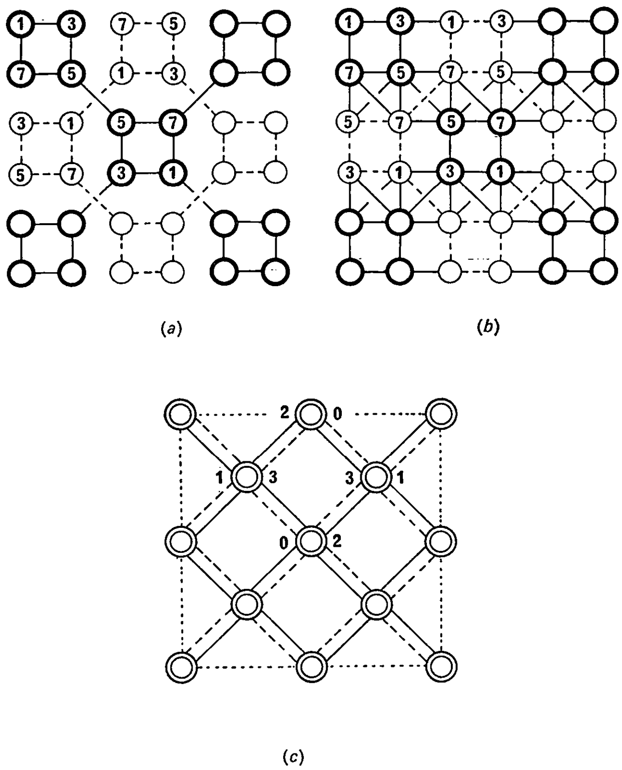


Fig. 4. Pairs of interpenetrating nets: (*a*) D- and L-forms of cubic (10,3), (*b*) two D- (10,3) nets, and (*c*) two diamond nets. In (*a*) and (*b*) numbers indicate heights as multiples of *c*/8 and in (*c*) as multiples of *c*/4.

points though not of *A* by 6*B* nor of *B* by 6*A*. This solution emerges from the following alternative approach.

Regarding *P*, *I*, and *F* lattices simply as 6-, 8-, and 12-connected nets, we examine how they may be broken down into two identical interpenetrating nets, or more generally, into two (or more) identical or non-identical nets. One case where the two component nets are different is of special interest in connection with the polyhedra of part VII.

The *I* net gives the two cases noted above, of which (*b*) is illustrated in Fig. 5(*a*). The *F* net may be broken down into various numbers of interpenetrating nets. Taking an eightfold cell (in which the coordinates of the points are those of the special position 32(*b*) in *Fd*3*c*) we may separate the 32 points into:

8 6-connected nets: 4(*a*) in $P4_13$;
4 3-connected nets: 8(*a*) in $I4_13$;

or

2 6-connected nets: 16(*c*) in *Fd*3, *Fd*3*m*, or $F4_13$.

(The formation of the *F* lattice by interpenetration of four 3-connected (10,3) nets may be of interest in connection with magnetic properties of alloys. It is interesting that there are two ways of interpenetrating four (10,3) nets, the individual (enantiomorphic) nets being either four D- (or L-) or two D- and two L- nets (Fig. 6).

The two 6-connected nets which interpenetrate to form the *F* lattice are basically of the diamond type. The equivalent points 16(*c*) in *Fd*3 *etc.* are the mid-points of the links of the diamond net, and when joined as in Fig. 7 they outline tetrahedra around the points of that net. The edges of the tetrahedra form a 6-connected net. Another way of breaking down the *F* lattice into two identical interpenetrating nets is shown in Fig. 5(*b*). The two nets are again of the diamond type, but the interbond angles are two of 90° and four of 120°. This 'pseudo-diamond net corresponds to the half-space-filling by rhombic dodecahedra (Fig. 8); the space-filling by rhombic dodecahedra corresponds, as is well known, to cubic closest packing of equal spheres (In Fig. 8 and subsequent stereo-pairs of 3-D polyhedra black faces indicate open ends of tunnels, or faces which will be shared in the infinite polyhedron).

Breakdown of the *P* lattice into two *different* nets is of interest in connection with complementary polyhedra [Fig. 5(*c*)]. The large cell contains 8 points, and the component nets have $Z=2$ (8-connected body-centred net) and $Z=6$ (planar 4-connected NbO net). This is the case to which reference has already been made in which a point *A* of the first net has 6 octahedral nearest *B* neighbours and *B* has $(2A+4B)$ octahedral nearest neighbours. Examples of pairs of complementary polyhedra related in this way are included in Table 2. Because of this relation between the *I* and NbO nets it follows that to every polyhedron based on the NbO net there is a complementary polyhedron built of identical 8-tunnel units based on the *I* net. The converse is not, however, true. If the 8-tunnel polyhedron has cubic symmetry there is a 4-tunnel complementary polyhedron built of identical repeat units. If on the other hand the symmetry of the 8-tunnel polyhedron is lower (*e.g.* the tetragonal (3,8) of Fig. VII.18* or the orthorhombic (3,7) of Fig. X.3†) there is no complementary polyhedron built of identical 4-tunnel units. The relation between the complementary 4-tunnel and 8-tunnel polyhedra in such cases will not be further discussed in the present paper.

It appears that all pairs of complementary 3-D polyhedra are based on one of the following pairs of interpenetrating nets:

(i) Two (4-connected) diamond (or 'pseudo-diamond') nets;

(ii) Two (6-connected) P nets; or

(iii) I (8-connected) and NbO (4-connected) nets.

In (i) and (ii) the two complementary polyhedra *may* be identical or they may be different; in (iii) the polyhedron and its complement are necessarily different in shape. Some examples are given in Table 2.

We shall see shortly that only a limited number of polyhedra can be built with all faces regular plane n-gons. In cases where this is possible it is important, when discussing the relation between the polyhedron and its complement, to distinguish between the polyhedron (n,p) built with curved (equilateral) faces and the topologically similar (n,p) built with plane equilateral faces. For example, Fig. VII.10 (*b*) and (*c*) showed repeat units of two (topologically different) 6-tunnel (3,8) polyhedra; they are repeated here as Fig. 9 (*b*) and (*c*). The unit (*b*) has six identical tunnels and the corresponding 3-D polyhedron is different from its complement. This particular polyhedron can be built with plane equilateral faces, when it is equivalent to a system of snub cubes (alternately D- and L-) joined through their square faces (Fig. 10). The complement is the regular 3-D polyhedron I_{6t} of Fig. 18 which has a quite different shape and volume, as in the case of the curved-face polyhedron and complement. On the other hand, the unit (*c*), which has the two polar tunnels larger than the four equatorial ones, forms a (curved-face) polyhedron which is identical to its complement (though the two are differently oriented), but this polyhedron cannot be built with regular plane faces.

* Fig. 18 of Part VII.
† Fig. 3 of Part X (present paper).

Regular and semi-regular 3-D polyhedra

From Euler's relation for convex simply-connected polyhedra, $N_0-N_1+N_2=2$, the following equations can be derived:

$$p=3:\ 3f_3+2f_4+f_5\mp 0f_6-f_7-\ \ldots=12 \tag{1}$$

$$p=4:\ 2f_3\mp 0f_4-2f_5-4f_6-6f_7-\ \ldots=16 \tag{2}$$

$$p=5:\ f_3-2f_4-5f_5-8f_6-11f_7-\ \ldots=20 \tag{3}$$

$$p=6:\ 0f_3-4f_4-8f_5-12f_6-\ \ldots=24 \tag{4}$$

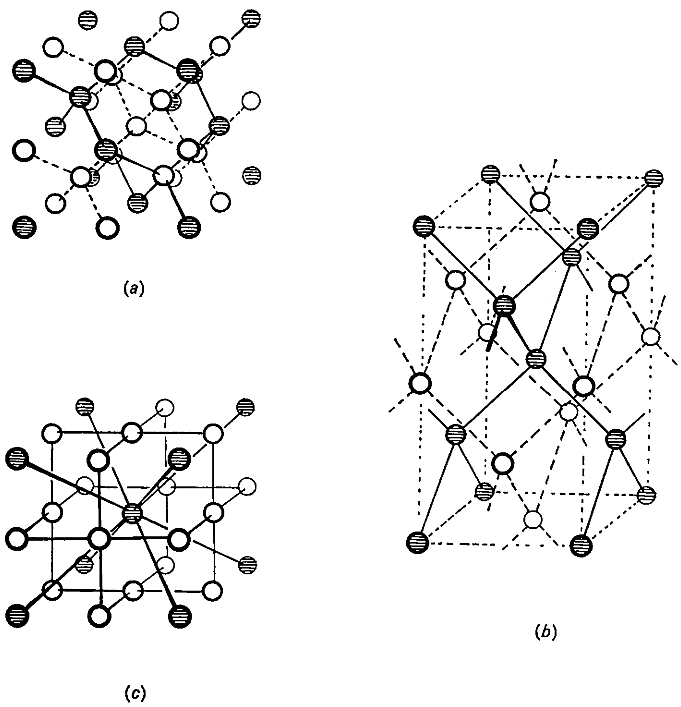


Fig. 5. (*a*) I net as two interpenetrating diamond nets, (*b*) F net as two interpenetrating pseudo-diamond nets, and (*c*) P lattice as interpenetrating I and NbO nets.

or more generally

$$\Sigma f_n[4-(2-n)(2-p)]=4p,$$

where f_n is the number of n-gon faces of which p meet at each vertex. It follows that there are only five finite polyhedra having all faces of the same kind and all vertices p-connected. The most symmetrical forms of these polyhedra are the regular (Platonic) solids:

3-connected: tetrahedron, (3,3), $f_3=4$, cube, (4,3), $f_4=6$,
and dodecahedron, (5,3), $f_5=12$;
4-connected: octahedron, (3,4), $f_3=8$; and
5-connected: icosahedron, (3,5), $f_3=20$.

These equations also show that there can be no 6-connected polyhedra; instead, there is the unique plane net (3,6).

It was shown in part VII that the form of Euler's relation relevant to the 3-D polyhedra is : $Z-N_1+N_2=2-t$, where Z is the number of points (vertices) in the *repeat unit* and t is the number of tunnels connecting each unit to its neighbours. Examination of the derivation of equations (1)–(4) (Wells, 1956) shows that the more general form of these equations is

$$\Sigma f_n[4-(2-n)(2-p)]=2p(2-t). \tag{5}$$

The number (Z) of points in the (topological) repeat unit is given by

$$Z=\frac{2n(2-t)}{4-(2-n)(2-p)}\,. \tag{6}$$

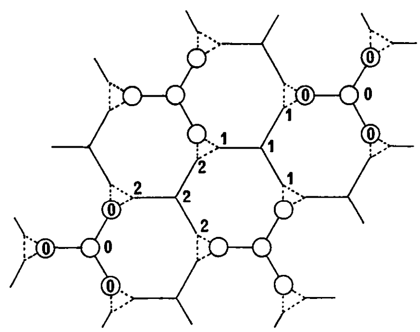
For convenience we give the expanded form of (5) for two values of p:

$$p=7:\ -\ f_3-6f_4-11f_5-16f_6-\ldots=14(2-t) \quad (7)$$
$$p=8:\ -2f_3-8f_4-14f_5-20f_6-\ldots=16(2-t). \quad (8)$$

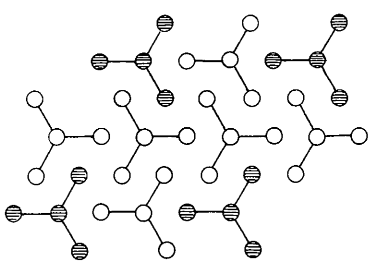
Such equations have positive roots for $t\geq 3$ and give the required relations between the numbers and types of polygons forming the surfaces of infinite 3-D polyhedra, f_n now being the number of n-gon faces in the repeat unit. Whereas each of the equations (1)–(3) has only one solution for f_3, each of the equations (7), (8), *etc.* has a set of solutions, one for each value of t. (Moreover there may be more than one way of constructing a polyhedron with specified values of n,p, and t – see Table 1.) We may therefore draw up a table for each value of n; Table 3 is for the triangulated polyhedra ($n=3$). There is no reason to suppose that all these polyhedra could be constructed. Our particular interest lies in those which are realizable with plane equilateral triangular faces, the 3-D homologues of the Platonic solids. We have accordingly approached this problem in the way described in the next section.

Equation (5) also gives more general forms of equations (1)–(4), for example:

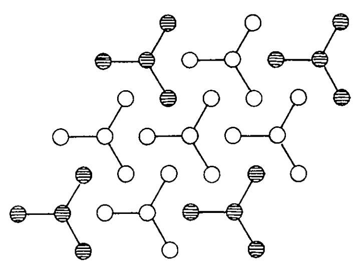
$$p=3:\ 3f_3+2f_4+f_5\mp 0f_6-f_7-2f_8-\ \ldots\ =6(2-t) \quad (1a)$$



(a)



(b)



(c)

Fig. 6. (*a*) Single cubic (10,3) net viewed along 3_1 axis. Numbers indicate heights of points above the plane of the paper as multiples of $c/3$, c being the repeat distance along the helix. (*b*) F net as four interpenetrating (10,3) nets, two D- and two L-. (*c*) F net as four interpenetrating (10,3) nets, all D- (or L-). The points shown in (*b*) and (*c*) are only those at one height (say, 0), the four groups of shaded circles belonging to one net.

which relates to the (3-connected) reciprocals of the triangulated polyhedra. For example, a 3-connected surface tessellation of 8-gons has $3(t-2)$ 8-gons in the repeat unit. There are also solutions of these equations corresponding to 3-D polyhedra with faces of more than one kind; these are the 3-D homologues of the Archimedean solids. Examples are (*a*) the polyhedron formed from truncated tetrahedra linked through octahedra to form a diamond-like structure having 3 triangular and 2 6-gon faces meeting at each vertex; (*b*) the diamond-like arrangement of truncated octahedra linked through hexagonal prisms to four other truncated octahedra (the basic net of faujasite), with 3 4-gons and 1 6-gon meeting at each vertex; (*c*) the 6-tunnel polyhedron formed from truncated octahedra linked through cubes (the net of zeolite *A*), with 2 4-gons and 2 6-gons meeting at each vertex.

Like the triangulated bodies these semi-regular 3-D polyhedra represent an extension of tessellations beyond those possible on the Euclidean plane. Systems

$p=$ 3	4	5
4^3	4^4	4^5
4^28	4^38	
$4\cdot8^2$	4^28^2	
8^3		

of 4- and 8-gons include 4^3 (cube) and 4^28 (octagonal prism), 4^4 and 4^28 (plane nets), and the higher members which are only realizable as 3-D polyhedra (to the right of the full line). The space-filling arrangement of truncated cuboctahedra (tco) and octagonal prisms (p_8) illustrates two semi-regular 3-D polyhedra, 4^28^2 and 4^36, and also the breakdown of a polyhedral space-filling into two different pairs of complementary polyhedra. The positions of the centres of the two kinds of polyhedra are those of the shaded circles (tco) and open circles (p^8) of Fig. 5(*c*). Each tco is in contact with eight others (sharing 6-gon faces) so that there is a continuous body-centred system of tco's. The octagonal prisms also form a continuous system by sharing alternate 4-gon faces to form a 4-tunnel 3-D polyhedron (4^28^2) based on the NbO net which is complementary to the polyhedron formed from the tco's. Alternatively, one half of the tco's and one half of the P^8's form a 6-tunnel 3-D polyhedron 4^36, based on the *P* lattice, which is identical with its complement (Fig. 11):

$$(\text{tco})_2(p_8)_6 \begin{cases} (\text{tco})_2 + (p^8)_6 & I \text{ net} \quad \text{NbO net} \\ \text{two } (\text{tco})(p_8)_3 & P \text{ net} \end{cases}$$

Other semi-regular 3-D polyhedra have all faces of the same kind (*e.g.* triangles) but vertices of more than one kind; these could be described as homologues of the Catalan solids. For example, the polyhedron formed from icosahedra joined through four icosahedra to form a diamond-like structure has equal numbers of 5- and 8-connected vertices, 12 of each in the repeat unit. The relevant equation for triangulated polyhedra is

$$3c_3+2c_4+c_5\mp0c_6-c_7-2c_8-3c_9-\ldots=6(2-t) \qquad (9)$$

which, like equation 1(*a*) is derived from the modified Euler equation. Evidently the choice of a particular

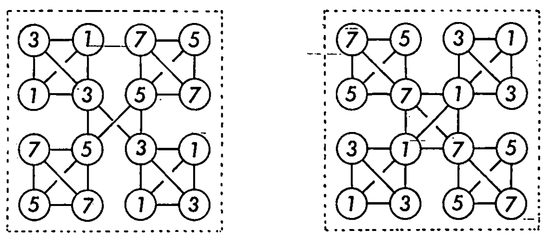


Fig. 7. Two 6-connected nets which together form an *F* net. Numbers indicate heights as multiples of $c/8$.

Table 2. *Some* 3-D *polyhedra and their complements*

Polyhedron and complement	Regular plane faces	Equilateral curved faces
IDENTICAL*		
(i) Diamond net	Skew polyhedron {6,6/3}	
Pseudo-diamond	Half-space-filling by rhombic dodecahedra X.8	
(ii) *P* net	Skew polyhedra {4,6/4} {6,4/4}	(3,8), X.9(*c*)
	(4^36), X.11	(3,9) VII.20(*a*)
		(3,10), VII.20(*b*)
Polyhedron and complement DIFFERENT		
(i) Diamond net	(3,9), X.13	
(ii) *P* net	(3,8), X.18	(3,8), X.9(*b*)
(iii) *I* and NbO nets	(3,9), X.19	(3,8), VII.15
	(4,5), X.23	(4,5), X.22

* The volume of a given 3-D polyhedron with regular *plane* faces is determined by the edge length only. This is also true of the curved-face polyhedra of part VII since they are constructed with *equilateral* faces. The various curvatures (and hence the volumes of the polyhedra) are not variable for a given tessellation on a given surface. Without the condition of equilateral faces the volume would be infinitely variable, since the surface of a polyhedron arises by inflating the links of a 3-D net.

combination of vertex types does not uniquely define the polyhedron. For example, solutions for 4-tunnel polyhedra having c_5 and c_8 vertices include

$$\begin{array}{llll} c_5: & 12 & 36 & 60 \\ c_8: & 12 & 24 & 36 \quad etc. \end{array}$$

which represent 3-D polyhedra having icosahedra at the points of the diamond net joined through tunnels consisting of respectively 1, 3, 5, *etc.* icosahedra forming linear tunnels by sharing a pair of opposite faces. Trivial solutions of any desired degree of complexity can be devised. The reciprocal tessellations corresponding to these triangulated polyhedra form further families of 3-connected nets including, for example, the polyhedron consisting of equal numbers of 5- and 8-gons – compare the plane net with equal numbers of 4- and 8-gons.

The 3-D homologues of the Platonic solids

It appears that a limited number of the polyhedra of Table 1 are realizable with plane regular faces. We consider here only the triangulated bodies. The triangulated Platonic solids have the following numbers of vertices: 4(tetrahedron), 6(octahedron), and 12(icosahedron). These solids may be placed at the points of the basic cubic nets and joined through tunnels which must also be bounded exclusively by equilateral triangles. If there is to be the same number of faces meeting at each vertex of the 3-D polyhedron the tunnels must not contain vertices other than those which are to be shared with the finite polyhedra placed at the points of the net. Therefore the tunnel must be either a triangulated polyhedron with 6 vertices and a pair of parallel faces (octahedron) which will act as a link between the polyhedra by sharing *one* face at each end with the finite polyhedra, or it may be a dodecahedral tunnel which can link at each end through *two* faces of the polyhedron. This tunnel is closely related to the octahedron from which it is derived by slitting along the edges accentuated in Fig. 12(*a*) and then simultaneously compressing along the directions AB and CD. If AB and CD are made equal in length to the other edges this object represents a group of five regular tetrahedra, one on each face of a central tetrahedron [Fig. 12(*b*)].

Since the octahedral tunnel involves one face of the central finite polyhedron and the dodecahedral tunnel two (adjacent) faces, and having regard to the symmetry of the central polyhedron, the systems to be studied are those set out in Table 4. All the polyhedra with octahedral tunnels can be constructed, but only one of those with dodecahedral tunnels. The shape of the latter tunnel can be adjusted within certain limits, but it is not possible to link octahedra through either three or four of these tunnels or *regular* icosahedra through six such tunnels. The dihedral angle of a regular icosahedron is 138° 12′, as compared with the value 148° 24′ of the angle DCE–DCF for the tunnel built of regular tetrahedra. However, a modified tunnel, with the eight exterior faces equilateral triangles and the shaded faces of Fig 12(*b*) isosceles, fits over pairs of adjacent faces of a 'cubic' icosahedron. This solid, with vertices $(\frac{1}{2}\frac{1}{4}0)$, $(\frac{1}{2}\frac{3}{4}0)$, *etc.*, has 8 equilateral faces (edge length $\frac{1}{2}\sqrt{6}$) and six pairs of isosceles faces with the common edge of unit length. (The dihedral

Table 3. *Number of faces in topological repeat units of* (3,*p*) *polyhedra*

p \ *t*	3	4	5	6	7	8	9	10	11	12
7	14	**28**	42	56	70	84	98	112	126	140
8	8	**16**	24	**32**	40	48	56	64	72	80
9	6	**12**	18	24	30	**36**	42	48	54	60
10	*	10	*	**20**	*	30	*	40	*	50
11	†	†	†	†	22	†	†	†	†	44
12	4	8	12	16	20	**24**	28	32	36	40

* Z non-integral.
† Number of faces non-integral.
Heavy type distinguishes polyhedra realizable with plane equilateral triangular faces (see Table 4).

Table 4. *The triangulated regular* 3-D *polyhedra*

Polyhedron at points of net	Z	Octahedral tunnels	Dodecahedral tunnels	p	f_3
Tetrahedron	4	4 T_{4t}		9	12
Octahedron	6	4 O_{4t}		8	16
		6 O_{6t}	(3)*	10	20
		8 O_{8t}	(4)*	12	24
Icosahedron	12	4 I_{4t}		7	28
			6 I_{6t}	8	32
		8 I_{8t}		9	36

* Not realizable.

angle over this edge is 126° 52′.) It is also necessary to check whether further 3-D polyhedra arise from Archimedean solids joined by sharing 4-gon, *etc.* faces. The only possibility appears to be the system of snub cubes (alternately D- and L-) to which reference has already been made. This polyhedron is in fact the complement of I_{6t}.

We have, therefore, 7 triangulated 3-D regular solids (and their complements), and of these the three 4-tunnel ones are related in a very simple way to the corresponding finite regular solids:

$p=$	3	4	5	6	7	8	9
	tetra-hedron	octa-hedron	icosa-hedron	(plane net)	I_{4t}	O_{4t}	T_{4t}
Number of vertices	4	8	12				
Z					12	8	4

Some points of interest concerning the seven triangulated polyhedra are now noted.

T_{4t} (3,9)–Fig. 13

This polyhedron, formed from tetrahedra at the points of the diamond net joined through octahedra along the links of that net, can obviously be built from octahedra only, each sharing 6 edges with other octahedra. If built of such AX_6 groups the composition is AX_2. The X atoms are in the positions of cubic closest packing and the structure is derived from the NaCl structure by removing the appropriate rows of A atoms. This is the idealized structure of atacamite, one of the polymorphs of $Cu_2(OH)_3Cl$.

O_{4t} (3,8)–Fig. 14

Octahedra at the points of the diamond net joined through octahedra along the links of that net form this polyhedron, which was illustrated in Fig. VII.12 as a polyhedron with curved faces.

O_{6t} (3,10)–Fig. 15

Octahedra at the points of a 6-connected net are connected through octahedral tunnels involving all but a pair of opposite faces. (This is a different (3,10) polyhedron from that of Fig. VII,20(*b*).)

O_{8t} (3,12)–Fig. 16

This polyhedron, built from octahedra joined through 8 octahedral tunnels, has been illustrated (with curved faces) in Fig. 2(*b*) of the present paper.

I_{4t} (3,7)–Fig. 17.

This is formed from icosahedra linked to four others through octahedra. The regular icosahedron may be referred to orthogonal axes which pass through the mid-points of opposite edges and are axes of twofold symmetry. The faces fall into two groups, 6 pairs and 8, arranged octahedrally. One half of the latter, a group of 4 tetrahedral faces, are those involved in the tunnels. When four such faces of a regular icosahedron are distinguished from the remainder (*e.g.* by colouring)

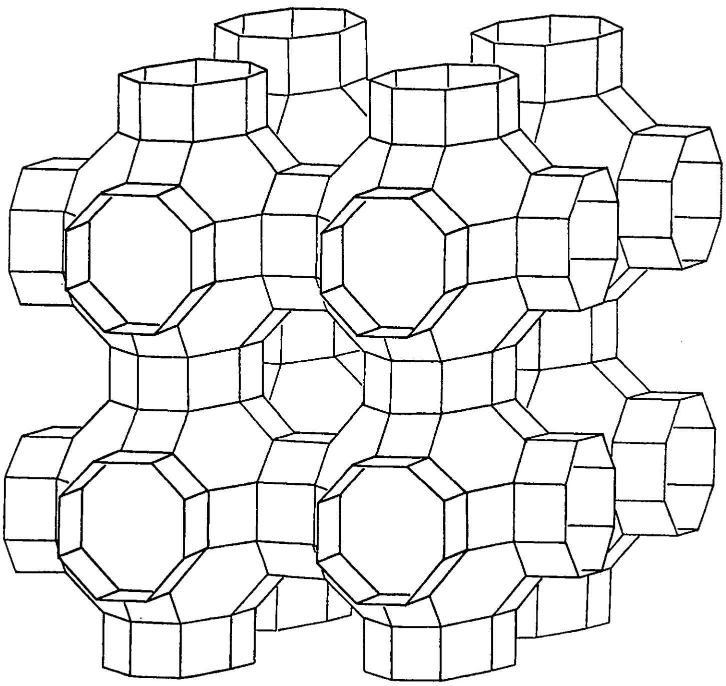


Fig. 11. Half-space-filling by equal numbers of truncated cuboctahedra and octagonal prisms.

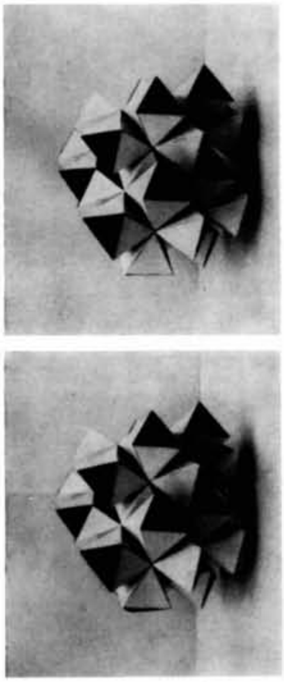


Fig. 16. O_{8t} (3,12).

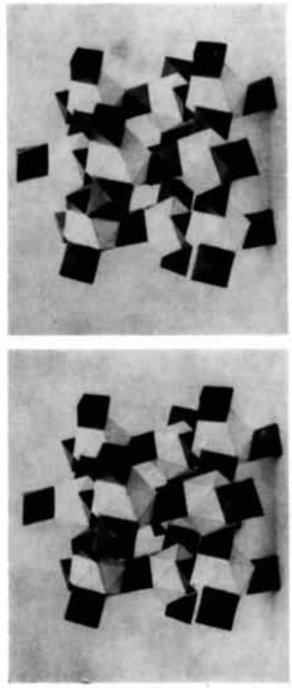


Fig. 17. I_{4t} (3,7).

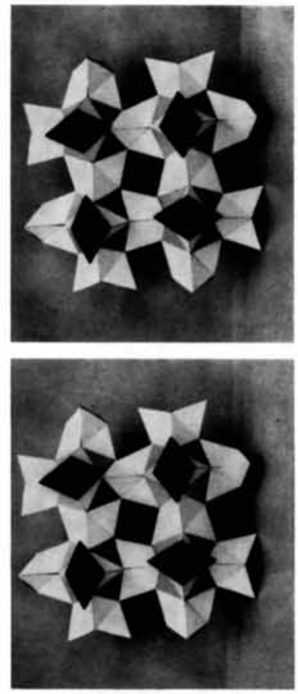


Fig. 18. I_{6t} (3,8).

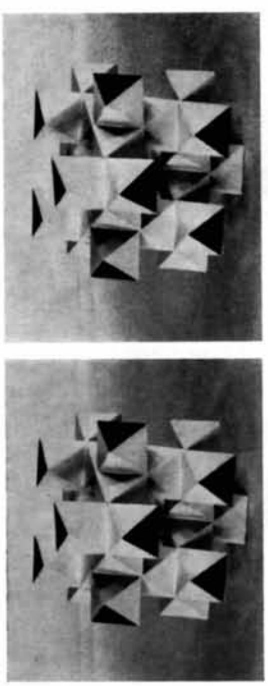


Fig. 13. T_{4t} (3,9).

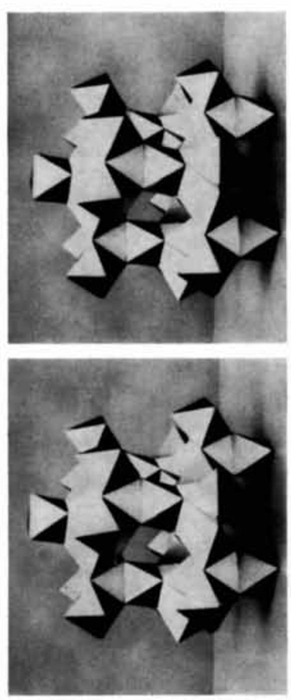


Fig. 14. O_{4t} (3,8).

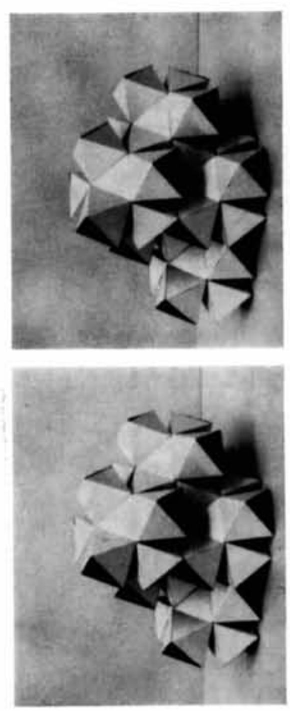


Fig. 15. O_{6t} (3,10).

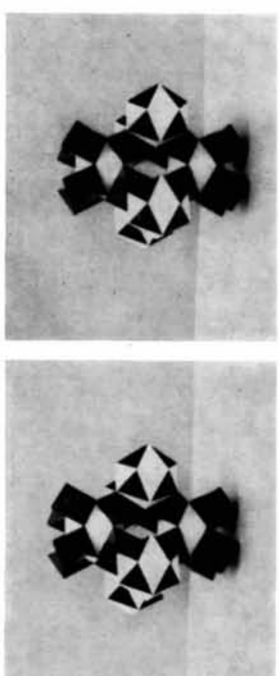


Fig. 19. I_{8t} (3,9).

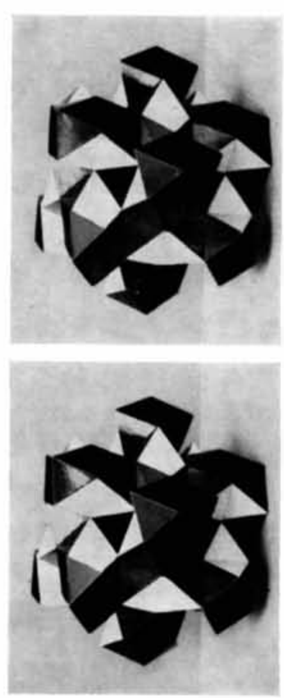


Fig. 20. I_{8t} (complement).

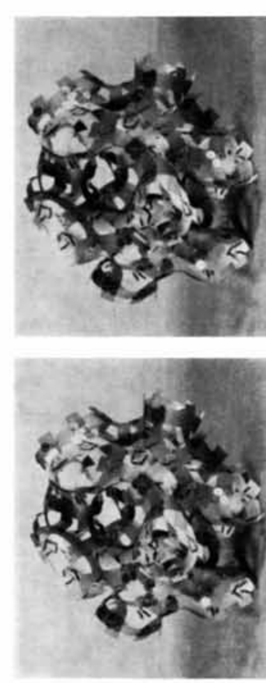


Fig. 21. 4t −(5,4).

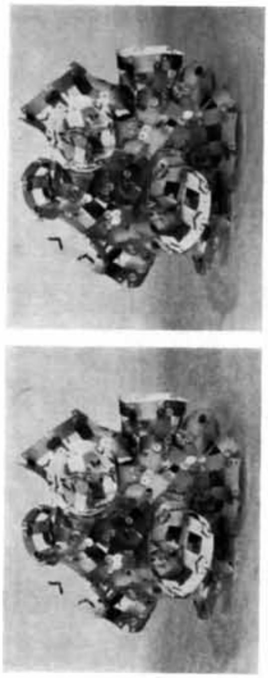


Fig. 22. 4t −(4,5).

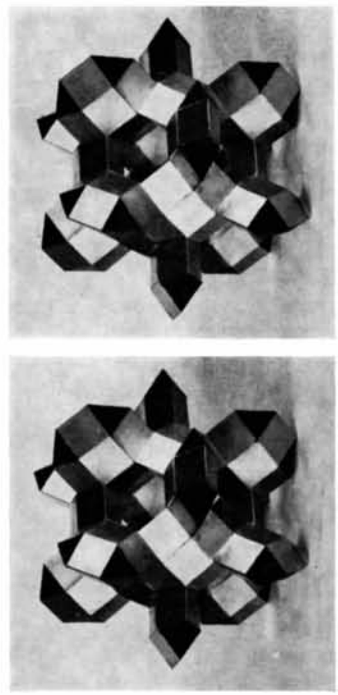


Fig. 23. Regular 3-D polyhedron (4,5).

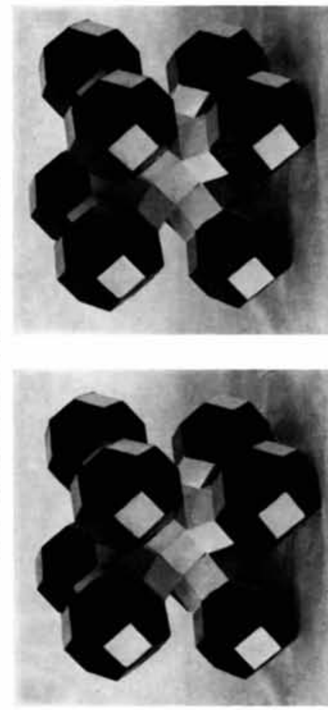


Fig. 24. (4,5) complement to Fig. 23.

the icosahedron is enantiomorphic. It is found by experiment that if all the icosahedra are D- or L- they do not join together to form a 3-D polyhedron. It is necessary to join alternate D- and L- icosahedra, as might be expected from the fact that there is a centre of symmetry at the mid-point of each link of the diamond net.

I_{6t} (3,8)–Fig. 18

This polyhedron, the only one incorporating dodecahedral tunnels, is built from 'cubic' icosahedra. The complementary polyhedron, an assembly of snub cubes, has been illustrated in Fig. 10 and with curved faces in Fig. 9(*b*).

I_{8t} (3,9)–Fig. 19

This polyhedron, built from icosahedra joined to 8 others through octahedral tunnels, has been illustrated with curved faces in Fig. 2(*a*). The octahedra form a connected system, each sharing all its vertices with 6 others, the system of octahedra therefore being topologically similar to that in the ReO_3 and FeF_3 structures. The complementary polyhedron, based on the NbO net, is shown in Fig. 20.

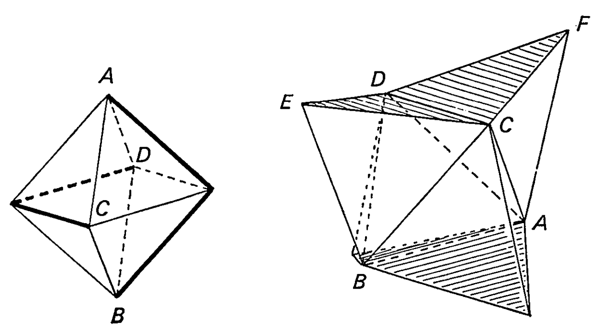
It may be found useful to summarize the contents of this paper and part VII as follows.

(i) On surfaces arising by inflating the links of 3-D nets various tessellations (n,p) may be inscribed. For given values of n and t there is apparently an upper limit for p. For example, the series $(3,p)$, $t=4$ (coplanar) extends from (3,7) to (3,12), while $(3,p)$ $t=3$ apparently comprises only (3,7) and (3,8) (see Table 1). As in the case of polyhedra and plane nets there is a tessellation (p,n) reciprocal to each (n,p). A reciprocal pair such as (4,5) and (5,4) are related in the same way as the icosahedron, (3,5), and dodecahedron (5,3).

(ii) In some cases topologically different tessellations having the same values of n and p may be constructed on surfaces derived from the same 3-D net — for example, the two (3,8) tessellations of Fig. 9 (*b*) and (*c*) based on the *P* net.

(iii) These 3-D polyhedra conform to the general equation

$$\Sigma\, f_n[4-(2-n)(2-p)]=2p(2-t)\,,$$



(*a*) (*b*)

Fig. 12. Relation of dodecahedral tunnel to octahedron (see text).

where t is the number of tunnels (the connectedness of the basic net). Solutions for $t=0$ correspond to the ordinary finite polyhedra, and those for $t\geq 3$ to infinite 3-D polyhedra. The significance of solutions for $t=1$ is not obvious, but $t=2$ corresponds to plane nets drawn on an infinite cylindrical surface. For example, if the right-hand side of equation (1) is put equal to zero there are solutions such as $f_5=f_7$, $3f_3=f_7$, *etc.*, which are equivalent to plane nets, $\varphi_5=\varphi_7=\frac{1}{2}$, $\varphi_3=\frac{1}{4}$, $\varphi_7=\frac{3}{4}$, *etc.*

(iv) Certain of these polyhedra can be constructed with plane equilateral faces; these are 3-D homologues of the Platonic solids and of the Archimedean and Catalan semi-regular solids. Only the triangulated bodies $(3,p)$ have been studied systematically, of which there are apparently seven.

(v) The space which is not occupied by a 3-D polyhedron may be described as the complementary polyhedron. In certain cases the polyhedron and its complement are identical, *i.e.* the surface divides space into two equal parts (Table 2). All the five polyhedra of this kind realizable with equilateral plane faces represent half-space-fillings by assemblies of finite convex polyhedra:

	Half-space-filling by:	3-D homologue of:
4^6	Cubes	Platonic solids
6^4	Truncated octahedra	
6^6	Tetrahedra and truncated tetrahedra	
$4^3 6$	Truncated cuboctahedra and octagonal prisms	Archimedean solids
$4.\frac{4}{8}$	Rhombic dodecahedra	Catalan solids

Two examples will illustrate the above relationships:

P net →	(3,8) curved faces →	reciprocal (8,3)	
	Fig. X.9(*b*)	Fig. VII.28	
	(3,8) plane faces →	complement (I_{6t})	
	Fig. X.10	Fig. X.17	
NbO net →	(4,5) curved faces →	reciprocal (5,4)	
	Fig. X.22	Fig. X.21	
	(4,5) plane faces →	complement ←	*I* net
	Fig. X.23	Fig. X.24	

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References

Wells, A. F. & Sharpe, R. R. (1963). *Acta Cryst.* **16**, 857.
Wells, A. F. (1956). *The Third Dimension in Chemistry*, p. 37. Oxford: Clarendon Press.